

Failures of the Silver Dichotomy in the Generalised Baire Space

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Abstract

We prove results that falsify Silver's dichotomy for Borel equivalence relations on the generalised Baire space under the assumption $V = L$.

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1 Introduction

The study of Borel equivalence relations and their reducibility springs from the interest in classification problems in mathematics. The classical theory studies Borel and analytic equivalence relations on Polish spaces and the partial order formed by these equivalence relation with respect to Borel reducibility $\langle \mathcal{E}, \leq_B \rangle$. The generalised descriptive theory, initiated in the 1990's by the work of Halko, Mekler, Väänänen and Shelah [Hal96, HS01, MV93], and recently developed further [FHK13, Kul13, Lüc12] studies the classification problems on generalised Baire and Cantor spaces, κ^κ and 2^κ for uncountable regular κ . As is already a custom we concentrate on cardinals with $\kappa^{<\kappa} = \kappa$.

We show that the classical result, known as the Silver dichotomy, fails in the generalised setting in the following two ways. It was shown in [Kul13], in particular, that the power set of κ ordered by inclusion, $\langle \mathcal{P}(\kappa), \subset \rangle$ can be embedded into $\langle \mathcal{E}, \leq_B \rangle$. In this paper we show that if κ is inaccessible and $V = L$, then $\langle \mathcal{P}(\kappa), \subset \rangle$ can be embedded into $\langle \mathcal{E}, \leq_B \rangle$ *below* the identity relation (Theorem 10). Then we show that if $V = L$ and κ is uncountable and regular, then there is an antichain with respect to \leq_B of length 2^κ of Borel equivalence relations and each of these relations is also incomparable with the identity relation.

In this paper we always work in $\text{ZFC} + V = L$ unless stated otherwise.

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2 Preliminaries

2.1 Fat Diamond

1 Definition (Fat diamond). A *fat diamond* on κ , denoted \diamond_κ , is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ such that for all $\alpha < \kappa$, $S_\alpha \subset \alpha$ and for every set $S \subset \kappa$, cub set $C \subset \kappa$ and $\gamma < \kappa$ there is a continuous increasing sequence of order type γ inside the set

$$C \cap \{\alpha < \kappa \mid S \cap \alpha = S_\alpha\}.$$

2 Theorem ($V = L$). A \diamond_κ -sequence exists for uncountable regular κ .

Proof. Suppose $\beta \leq \kappa$, $(S_\alpha)_{\alpha < \beta}$ is defined and (C^*, S^*, γ^*) is a triple such that $C^*, S^* \subset \beta$, $\gamma^* < \beta$, C^* is cub in β and there exists no continuous increasing sequence of order type γ^* in $C^* \cap \{\alpha \mid S^* \cap \alpha = S_\alpha\}$. Then we abbreviate this by $R(C^*, S^*, \gamma^*)$.

Let $(C_0, S_0) = (\emptyset, \emptyset)$. By induction, suppose that (C_α, S_α) is defined for all $\alpha < \beta$. Let (C_β, S_β) be the L -least pair such that for some $\gamma < \beta$ we have $R(C_\beta, S_\beta, \gamma)$, if such exists and set $(C_\beta, S_\beta) = (\emptyset, \emptyset)$ otherwise.

Let us show that the sequence $\langle S_\alpha \rangle_{\alpha < \kappa}$ obtained in this way is a \diamond_κ -sequence. Note that this sequence is definable in L . Suppose on the contrary that it is not a \diamond_κ -sequence. Then there exists a cub C and $S \subset \kappa$ such that there exists an ordinal $\gamma < \kappa$ with $R(C, S, \gamma)$. Suppose that (C, S) is the L -least such pair and γ the least ordinal witnessing this. Note that then (C, S) and γ are definable in L . Then build a continuous increasing sequence $(M_\beta)_{\beta < \gamma}$ of elementary submodels of L_{κ^+} such that $(M_\beta \cap \kappa)_{\beta < \gamma}$ is a continuous increasing sequence of ordinals in C . Let us show that each $M_\beta \cap \kappa$ is also in $\{\alpha \mid S \cap \alpha = S_\alpha\}$ which is a contradiction.

So let $\beta < \gamma$, denote $\beta' = M_\beta \cap \kappa$ and let π be the transitive collapse of M_β onto some $L_{\beta''}$. Then $\pi(C) = C \cap \beta'$ and $\pi(S) = S \cap \beta'$. Moreover by the elementarity of M_β in L

$$M \models (C, S) \text{ is the } L\text{-least pair s.t. } \exists \delta < \kappa R(C, S, \delta).$$

Applying π we get

$$L_{\beta''} \models (C \cap \beta', S \cap \beta') \text{ is the } L\text{-least pair s.t. } \exists \delta < \beta' R(C \cap \beta', S \cap \beta', \delta)$$

But then by absoluteness of the L -ordering, this holds also in L , so in fact $S \cap \beta' = S_{\beta'}$, so β' is in $\{\alpha \mid S \cap \alpha = S_\alpha\}$ as intended. \square

3 Definition. A stationary set $S \subset \kappa$ is *fat*, if for all cub sets $C \subset \kappa$ and all $\gamma < \kappa$ there is a continuous increasing sequence of length γ in $S \cap C$.

4 Theorem ($V = L$). *If $\kappa > \omega$ is regular, then there exists a fat stationary set $S \subset \kappa$ such that $\kappa \setminus S$ is also fat stationary.*

Proof. Let $S = \{\alpha \mid S_\alpha = \alpha\}$ where $(S_\alpha)_{\alpha < \kappa}$ is the \diamond_κ -sequence defined above. Then S is fat by definition. But also $S' = \{\alpha \mid S_\alpha = \emptyset\}$ is fat stationary and is disjoint from S . \square

2.2 Trees

5 Definition. $\kappa^{<\kappa}$ is the tree consisting of all functions $p: \alpha \rightarrow \kappa$ for $\alpha < \kappa$ ordered by end-extension: $p < q \iff p \subset q$. By a *tree* we mean a downward closed suborder of $\kappa^{<\kappa}$. A subtree is a subset of a tree which is itself a tree. Let $T \subset \kappa^{<\kappa}$ be a tree. A *branch* through T is a set b which is a maximal linear suborder of T . The set of all branches of T is denoted by $[T]$. The set of all branches of length κ is denoted by $[T]_\kappa$. The *height* of an element $p \in T$, denoted $\text{ht}(p)$, is the order type of $\{q \in t \mid q < p\}$. Let $\alpha < \kappa$ be an ordinal. Denote by T^α the subtree of T formed by all the elements with $\text{ht}(p) < \alpha$.

3 For an Inaccessible κ .

In this section we show that, if $V = L$ and κ is a strongly inaccessible cardinal, then there exists an embedding F of $\langle \mathcal{P}(\kappa), \subset \rangle$ into $\langle \mathcal{E}, \leq_B \rangle$ where \mathcal{E} is the set of Borel equivalence relations on 2^κ such that for all $A \in \mathcal{P}(\kappa)$, $F(A) \leq_B \text{id}_{2^\kappa}$.

6 Definition. Let $\text{sing}_\omega(\kappa)$ be the set of singular ω -cofinal cardinals below κ . We will construct for each set $S \subset \text{sing}_\omega(\kappa)$ a *weak S -Kurepa tree* T_S as follows. For each $\alpha \in \text{sing}_\omega(\kappa)$ let $f(\alpha)$ be the least limit ordinal β such that $L_\beta \models (\alpha \text{ is singular})$. Then let

$$T_S = \{s \in 2^{<\kappa} \mid \forall \alpha \leq \text{dom}(s) (\alpha \in S \rightarrow s \restriction \alpha \in L_{f(\alpha)})\}.$$

7 Lemma. *Let $S \subset \text{sing}_\omega(\kappa)$. Then for every $\gamma \in S$ we have $|T_S^{\gamma+1}| = |\gamma|$.*

Proof. When $\gamma \in S$, then $T_S^{\gamma+1}$ is a subset of $L_{f(\gamma)}$ whose cardinality is $|f(\gamma)|$. But $|f(\gamma)| < |\gamma|^+$, so we are done. \square

Remark. For the converse, if S is as in Lemma 7 and $\gamma \in S_\omega^\kappa \setminus S$, then we have $|T_S^{\gamma+1}| = 2^\gamma$: by GCH and $\text{cf } \gamma = \omega$, we have $|2^\gamma| = |\gamma^\omega|$ and it is enough to consider increasing cofinal sequences in γ . But if s is an increasing cofinal sequence in γ of length ω , then it is an element of T_S .

8 Lemma. *Let $T = T_S$ be a tree as above for some $S \subset \text{sing}_\omega(\kappa)$. Let $(D_i)_{i < \kappa}$ be a sequence of dense open subsets of T , i.e. such that for all $i < \kappa$ we have $\forall p \in T \exists q \in D_i (q > p)$ (density) and $\forall p \in D_i (N_p \cap T \subset D_i)$ (openness). Then there is a branch of length κ in $\bigcap_{i < \kappa} D_i$ through T .*

Proof. Suppose there is a sequence $(D_i)_{i < \kappa}$ of dense open subsets of $[T_S]_\kappa$ such that $\bigcap_{i < \kappa} D_i$ is empty. Suppose $(D_i)_{i < \kappa}$ is the L -least such sequence. For a contradiction it is enough to find a branch through $T_{\text{sing}_\omega(\kappa)}$ in $\bigcap_{i < \kappa} D_i$. Let $(M_\gamma)_{\gamma < \kappa}$ be a definable continuous increasing sequence of sufficiently elementary submodels of (L_{κ^+}, \in) of size $< \kappa$ such that $M_\gamma \cap \kappa = \gamma'$ for some $\gamma' < \kappa$ and M_γ contains a Borel code for D_γ . Let $L_{\gamma''}$ be the result of the transitive collapse of M_γ . Now pick the L -least $p_0 \in L_0''$ such that $L_0'' \models (N_{p_0} \subset D_0 \wedge p_0 \in T_{\text{sing}_\omega(\kappa)})$. Note that this implies that $p_0 \in T_{\text{sing}_\omega(\kappa)}$. If p_γ is defined to be an element of $L_{\gamma''}$, let $p_{\gamma+1}$ be the L -least element of $L_{(\gamma+1)''} \cap T_S$ extending p_γ such that $L_{(\gamma+1)''} \models (N_{p_{\gamma+1}} \subset D_{\gamma+1} \wedge p_{\gamma+1} \in T_{\text{sing}_\omega(\kappa)})$ and $\text{dom } p_{\gamma+1} > \gamma'$. If γ is a limit and p_β are defined for all $\beta < \gamma$, then let $p_\gamma = \bigcup_{\beta < \gamma} p_\beta$. The sequence $(M_\beta)_{\beta < \gamma}$ is definable in $L_{\gamma''}$, and so is p_γ . On the other hand $\text{dom } p = \gamma'$ is regular from the viewpoint of $L_{\gamma''}$, so $p_\gamma \in T_{\text{sing}_\omega(\kappa)}$. In this way we obtain a branch through $T_{\text{sing}_\omega(\kappa)} \subset T_S$ in $\bigcap_{i < \kappa} D_i$. \square

9 Lemma. *For every $S \subset \text{sing}_\omega(\kappa)$, T_S has κ^+ branches of length κ , i.e. $|[T_S]_\kappa| = \kappa^+$.*

Proof. As remarked above, $T_{\text{sing}_\omega(\kappa)} \subset T_S$, so it is sufficient to show that $T_{\text{sing}_\omega(\kappa)}$ has κ^+ branches. For each $\beta < \kappa^+$ let $C(\beta) = \{\gamma < \kappa \mid \text{SH}^{L_\beta}(\gamma \cup \{\kappa\}) \cap \kappa = \gamma\}$.

We want to show that there is an unbounded set $G \subset \kappa^+$ such that for all $\beta, \beta' \in G$ the sets $C(\beta)$ and $C(\beta')$ are all different if $\beta \neq \beta'$ and that the characteristic function of each $C(\beta)$ is a branch through $T_{\text{sing}_\omega(\kappa)}$. We claim that $G = \{\beta < \kappa^+ \mid \text{SH}^{L_\beta}(\kappa \cup \{\kappa\}) = L_\beta\}$ is such a set. To show that G is unbounded, let $\beta < \kappa^+$ and let $X \subset \kappa$ be a set such that $X \notin L_\beta$. Let $\beta' < \kappa^+$ be the least ordinal such that X is definable in $L_{\beta'}$ with parameters, so $\beta' \geq \beta$. Let $\varphi(p)$ be a formula with parameters p , which defines X and let p_0 be the L -least sequence of parameters such that $\varphi(p_0)$ defines a subset of $L_{\beta'}$ which is not an element of $L_{\beta'}$. Now p_0 is in $\text{SH}^{L_{\beta'}}(\kappa \cup \{\kappa\})$. Let $\bar{\beta}$ be such that $\text{SH}^{L_{\beta'}}(\kappa \cup \{\kappa\}) \cong L_{\bar{\beta}}$. We want to show that $\bar{\beta} = \beta'$. But since $p_0 \in \text{SH}^{L_{\beta'}}(\kappa \cup \{\kappa\})$, the set defined by

$\varphi(p_0)$ in $L_{\bar{\beta}}$ is in $L_{\bar{\beta}+1}$. But by the definition of p_0 , this set cannot be in $L_{\beta'}$, so $\bar{\beta} = \beta'$.

Suppose $\beta, \beta' \in G$ and $\beta < \beta'$. We claim that $C(\beta') \subset^* \lim C(\beta)$, where \subset^* means inclusion modulo a bounded set and \lim denotes the limit points of a set. This clearly implies that $C(\beta) \neq C(\beta')$. Suppose $\gamma \in C(\beta')$. Then since $\beta' \in G$, we have $\beta \in \text{SH}^{L_{\beta'}}(\gamma \cup \{\kappa\})$ for any γ greater than some $\gamma^* < \kappa$. Now every Skolem function of L_β is definable in $L_{\beta'}$ with parameters from $\gamma \cup \{\kappa\}$, so $\beta' \in C(\beta)$. But in fact, also $C(\beta)$ is definable in $L_{\beta'}$ with these parameters, so in fact $\beta' \in \lim C(\beta)$. Thus $C(\beta') \setminus \gamma^* \subset \lim C(\beta)$.

Let f_β be the characteristic function of $C(\beta)$ and let us show that $(f_\beta \upharpoonright \alpha)_{\alpha < \kappa}$ is a branch of $T = T_{\text{sing}_\omega(\kappa)}$. By the definition of T it is sufficient to show that $f_\beta \upharpoonright \alpha$ is in $L_{f(\alpha)}$ for all singular $\alpha \in \kappa$; this is of course equivalent to $C(\beta) \cap \alpha$ being in $L_{f(\alpha)}$. There are two cases: either $\alpha \in C(\beta)$ or $\alpha \notin C(\beta)$. If α is in $C(\beta)$, then by the definition of $C(\beta)$, $\text{SH}^{L_\beta}(\alpha \cup \{\kappa\}) \cap \kappa = \alpha$. Let $\bar{\beta}$ be such that $L_{\bar{\beta}}$ is the transitive collapse of $\text{SH}^{L_\beta}(\alpha \cup \{\kappa\})$. Then $C(\beta) \cap \alpha \in L_{\bar{\beta}+2}$ and since κ becomes α in the collapse, α is regular in $L_{\bar{\beta}}$ and so $\bar{\beta} < f(\alpha)$. But $f(\alpha)$ was chosen to be a limit ordinal, $\bar{\beta} + 2 < f(\alpha)$ as well. Thus $C(\beta) \cap \alpha \in L_{f(\alpha)}$. Suppose that $\alpha \notin C(\beta)$. But then $C(\beta) \cap \alpha$ is bounded in α and since α is a cardinal, $C(\beta) \cap \alpha \in L_\alpha \subset L_{f(\alpha)}$. \square

10 Theorem. *Suppose $V = L$ and κ is inaccessible. Then the order $\langle \mathcal{P}(\kappa), \subset \rangle$ can be embedded into $\langle \mathcal{E}, \leq_B \rangle$ (Borel equivalence relations) strictly below the identity on 2^κ . More precisely, there exists $F: \mathcal{P}(\kappa) \rightarrow \mathcal{E}$ such that for all $A_0, A_1 \subset \mathcal{P}(\kappa)$ we have $A_0 \subset A_1 \iff F(A_0) \leq_B F(A_1)$ and $F(A_0) \not\leq_B \text{id}_{2^\kappa}$.*

Proof. For a tree $T \subset 2^{<\kappa}$ let $E(T)$ be the equivalence relation on 2^κ such that two elements are equivalent if and only if both of them are not branches of T or they are identical.

10.1 Claim. *Suppose $S_0 \subset \kappa$ is a fat stationary set such that $S_\omega^\kappa \setminus S_0$ is stationary. (Such sets S_0 exist by Theorem 4.) Then if S' and S are stationary subsets of $S_\omega^\kappa \setminus S_0$ such that $S' \setminus S$ is stationary, we have $E(T_S) \not\leq_B E(T_{S'})$.*

Proof. Suppose to the contrary that $f: 2^\kappa \rightarrow 2^\kappa$ is a Borel reduction from $E(T_S)$ to $E(T_{S'})$.

The space $[T_S]_\kappa$ is equipped with the subspace topology inherited from 2^κ and we can define Borel, meager and co-meager subsets of $[T_S]_\kappa$. Note that the meager and co-meager subsets of $[T_S]_\kappa$ do not coincide with those in 2^κ , for example $[T_S]_\kappa$ is not meager in $[T_S]_\kappa$ by Lemma 8 but meager in 2^κ . Now we can define the Baire property relativised to $[T_S]_\kappa$: a set $A \subset [T_S]_\kappa$ has the Baire property, if there exists open $U \subset [T_S]_\kappa$ such that $U \triangle [T_S]_\kappa$ is meager in $[T_S]_\kappa$.

A standard proof gives that all Borel sets of $[T_S]_\kappa$ have the Baire property. For every $p \in T_{S'}$, the inverse image of N_p under f is Borel and so there is open U_p such that $U_p \triangle f^{-1}N_p$ is meager. Now let $D = [T_S]_\kappa \setminus \bigcup_{p \in 2^{<\kappa}} U_p \triangle f^{-1}N_p$. By Lemma 8 an intersection of κ many dense open sets is non-empty in $[T_S]_\kappa$ whereas it follows that the space is co-meager in itself and D is co-meager. So $D \subset [T_S]_\kappa$ is dense and f is continuous on D .

By removing one point from D , we may assume without loss of generality that $f: [T_S]_\kappa \rightarrow [T_{S'}]_\kappa$.

Let $c(S_0)$ be the set of increasing continuous sequences $(\alpha_\beta)_{\beta < \gamma}$ in S_0 with the property that also $\sup_{\beta < \gamma} \alpha_\beta \in S_0$. We will now define a function

$$\tau: c(S_0) \rightarrow \kappa \times \mathcal{P}(T_S) \times \mathcal{P}(T_{S'})$$

by induction on the length of the sequence $(\alpha_\beta)_{\beta < \gamma} \in c(S_0)$. The projection of τ to the first coordinate, $\text{pr}_1 \circ \tau$ can be thought as a strategy of a player in a climbing game (where the players pick ordinals below κ in an increasing way). Let $\tau(\emptyset) = (0, \{\eta \restriction \delta \mid \delta < \kappa\}, \{f(\eta) \restriction \delta \mid \delta < \kappa\})$ where η is any element of $[T_S]_\kappa \cap D$ and suppose $\tau((\alpha_\delta)_{\delta < \gamma+1})$ is defined to be (α, A, A') such that A and A' are subtrees of T_S and $T_{S'}$ respectively such that

1. $\alpha > \alpha_\delta$ for all $\delta < \gamma + 1$,
2. all the branches of A and A' have length κ ,
3. for each branch η of A , $N_{\eta \restriction \alpha} \cap [A]_\kappa = \{\eta\}$, i.e. there are no splitting nodes above α , and the same for A' ,
4. for each branch η of A , we have $f[D \cap N_{\eta \restriction \alpha}] \subset N_{\xi \restriction \alpha_\gamma}$ for some unique branch ξ of A' .

Note that the last condition defines an embedding from $[A]_\kappa$ to $[A']_\kappa$. Now we want to define $\tau((\alpha_\delta)_{\delta \leq \gamma+1}) = (\beta, B, B')$ where $\alpha_{\gamma+1} \in S_0 \setminus (\alpha_\gamma + 1)$. For each branch η of A , there is a branch ξ_η in $[T_S]_\kappa \cap D \cap N_{\eta \restriction \alpha_{\gamma+1}}$ such that $\xi_\eta(\alpha_{\gamma+1} + 1) \neq \eta(\alpha_{\gamma+1} + 1)$ (for example find ξ_η as follows: first note that the function ξ'_η such that $\xi'_\eta(\delta) = \eta(\delta)$ for $\delta \leq \alpha_{\gamma+1}$ and $\xi'_\eta(\delta) = 1 - \eta(\delta)$ for $\delta > \alpha_{\gamma+1}$ is a branch of T_S (because η is a branch and $\xi'_\eta \restriction \beta$ is definable from $\eta \restriction \beta$ for all β), so by the density of D , there is $\xi_\eta \in D \cap N_{\xi'_\eta \restriction \alpha_{\gamma+1} + 1}$). By condition (3) this branch is new (i.e. not in A). Let B be the downward closed subtree of T_S such that $[B]_\kappa = \bigcup_{\eta \in A} \{\eta, \xi_\eta\}$ and B' the same for $T_{S'}$ such that $[B']_\kappa = \{f(\eta) \mid \eta \in B\}$. Then pick β high enough so that condition (4) is satisfied for α , A and A' replaced by β , B and B' which is possible by the continuity of f on D .

Suppose γ is a limit and $(\alpha_\delta)_{\delta < \gamma}$ is in $c(S_0)$ and $\tau((\alpha_\delta)_{\delta < \varepsilon}) = (\beta_\varepsilon, B_\varepsilon, B'_\varepsilon)$ is defined for all $\varepsilon < \gamma$. Let us define $\tau((\alpha_\varepsilon)_{\varepsilon < \gamma}) = (\beta, B, B')$. Let β be the supremum of $\{\beta_\varepsilon \mid \varepsilon < \gamma\}$. Note that $\bigcup_{\varepsilon < \gamma} B_\varepsilon$ is a downward closed subset of T_S . Let p be any branch of length β through $\bigcup_{\varepsilon < \gamma} B_\varepsilon \cap 2^{<\beta}$. If there is a branch in $\bigcup_{\varepsilon < \gamma} B_\varepsilon$ that continues p , let $\eta(p)$ be that branch. Otherwise, since $\beta \notin S$ and $p \upharpoonright \gamma \in T_S$ for all $\gamma < \beta$, p can be continued to some branch η in $D \cap T_S$ and we define $\eta(p)$ to be that η . Let

$$B = \{\eta(p) \mid p \text{ is a branch through } \bigcup_{\varepsilon < \gamma} B_\varepsilon\}$$

and

$$B' = \{f(\eta) \mid \eta \in B\}.$$

Let C be the cub set of ordinals α that are closed under τ , in the sense that C is the set of those α such that for all sequences $s \in c(S_0)$ that are bounded in α , we have $(\text{pr}_1 \circ \tau)(s) < \alpha$, $|(\text{pr}_2 \circ \tau)(s)| < \alpha$ and $|(\text{pr}_3 \circ \tau)(s)| < \alpha$. For each pair of ordinals $(\alpha_1, \alpha_2) \in \kappa$, let $\pi(\alpha_1, \alpha_2)$ be the least ordinal such that there is an increasing continuous sequence of order type α_1 starting above α_2 with supremum at most $\pi(\alpha_1, \alpha_2)$ and let C_1 be the cub set of ordinals closed under π . Now by the stationarity of $S' \setminus S$, pick $\alpha \in C \cap C_1 \cap S' \setminus S$. Now it is easy to construct a continuous increasing sequence s in $c(S_0)$ of order type α , cofinal in α , and a cofinal sequence $(\gamma_n)_{n < \omega}$ in α such that $s \upharpoonright \gamma_n$ is in $c(S_0)$ for all n and $\tau(s \upharpoonright \gamma_n) = (\delta_n, A_n, A'_n)$ has the following properties:

- $\gamma_n \leq \delta_n < \alpha$,
- A_n has at least 2^{γ_n} many branches and
- f defines a bijection between the branches of A_n and the branches of A'_n

Let A_ω be the tree which consists of those branches η of T_S in D that for every $\delta < \alpha$ there is a branch ξ in $\bigcup_{n < \omega} [A_n]_\kappa$ such that the common initial segment of η and ξ has height at least δ . Since $\alpha \notin S$, the number of branches of A_ω is 2^α . So it means that the set $f[[A_\omega]_\kappa]$ must have 2^α branches too. The contradiction will follow once we show that this implies that $T_{S'}^{\alpha+1}$ must have 2^α elements, contradicting Lemma 7, because $\alpha \in S'$. But if η and ξ are any two branches in A_ω , their images must disagree below α by the construction, hence $T_{S'}^{\alpha+1}$ should have at least the same cardinality as $|A_\omega|$. \square

10.2 Claim. *If $S \subset S' \subset \kappa$, then $E(T_{S'}) \leq_B E(T_S)$.*

Proof. By the assumption we have $T_{S'} \subset T_S$. Let $i: [T_{S'}]_\kappa \rightarrow [T_S]_\kappa$ be the inclusion map and let ξ be a fixed element of $2^\kappa \setminus [T_S]_\kappa$. For $\eta \in 2^\kappa$, let $f(\eta) = i(\eta)$, if $\eta \in [T_{S'}]_\kappa$ and $f(\eta) = \xi$ otherwise. \square

To prove the Theorem, let S_0 be a fat stationary set such that $S_\omega^\kappa \setminus S_0$ is stationary. Let $\{S_i \mid i < \kappa\}$ be a partition of $S_\omega^\kappa \setminus S_0$ into κ many disjoint stationary sets. Then by the claims above, the function defined by

$$A \mapsto E(T_{\bigcup_{i \notin A} S_i})$$

is an embedding F of $\langle \mathcal{P}(\kappa), \subset \rangle$ into $\langle \mathcal{E}, \leq_B \rangle$ such that for all $A \in \mathcal{P}(\kappa)$, we have $F(A) \leq_B \text{id}_{2^\kappa}$ and by the same argument as in the proof of Claim 10.1, we have $\text{id}_{2^\kappa} \not\leq_B F(A)$ \square

4 An Antichain Containing the Identity

In this section κ is regular and uncountable, but not necessarily inaccessible. We now redefine the meaning of $\text{sing}_\omega(\kappa)$ to be the set of all ω -cofinal ordinals below κ (instead of just cardinals as in the previous section). Let $T = T_{\text{sing}_\omega(\kappa)}$ (see Definition 6).

As in Lemma 8, T is not meager in itself and we can define the ideal of meager sets relativised to T . In this way, the Borel subsets of T will have the Baire property in T . Note that T is a meager subset of 2^κ , so the meager ideal on subsets of T is not a straightforward restriction of the meager ideal on the subsets of 2^κ .

11 Lemma. *Suppose $f: T \rightarrow 2^\kappa$ is a Borel function. Then there is a co-meager set $D \subset T$ such that f is continuous on D .*

Proof. Using Lemma 8 as in the beginning of the proof of Claim 10.1 \square

12 Theorem. *Suppose $V = L$. Then there is an antichain of Borel equivalence relations with respect to \leq_B of size 2^κ such that one of the relations is the identity.*

Proof. Let $[T]_\kappa$ be the set of branches of length κ of T . Let $S \subset \kappa$ be stationary. Then let η and ξ be F_S -equivalent, either if both η and ξ are not in $[T]_\kappa$, or if both η and ξ are in $[T]_\kappa$ and are E_S -equivalent, where E_S is as in [Kul13]: η and ξ are E_S equivalent if they are E_0 -equivalent and for every $\alpha \in S$ there exists $\beta < \alpha$ such that $\forall \gamma \in [\beta, \alpha[, |\eta(\gamma) - \xi(\gamma)| = |\eta(\beta) - \xi(\beta)|$.

For a tree $T \subset 2^{<\kappa}$ and a stationary $S \subset \kappa$ define the following game $G(T, S)$ of length ω for two players **I** and **II**: At move $n < \omega$, player **I** picks a pair (p_n^0, p_n^1)

of elements of T with $\text{dom } p_n^0 = \text{dom } p_n^1$ and then player **II** picks an ordinal α_n above $\text{dom } p_n^0$. Additionally the following conditions should be satisfied by the moves of player **I**:

1. $p_{n-1}^0 \subset p_n^0$ and $p_{n-1}^1 \subset p_n^1$,
2. $\text{dom } p_n^0 = \text{dom } p_n^1 > \alpha_{n-1}$.

Suppose that $(p_n^i)_{n < \omega}$ for $i \in \{0, 1\}$ are the sequences obtained in this way by player **I**. Player **II** wins, if player **I** didn't follow the rules, or else $\bigcup_{n < \omega} p_n^0$ and $\bigcup_{n < \omega} p_n^1$ are both in T and $\sup_{n < \omega} \text{dom } p_n^0 \in S$.

12.1 Claim. *Suppose $S \subset \text{sing}_\omega(\kappa)$ is stationary and T is the weak Kurepa tree defined above. Then Player **I** has no winning strategy in $G(T, S)$.*

Proof. Suppose τ is a strategy of Player **I**. Let M be an elementary submodel of $(L_{\kappa^+}, \tau, S, \in)$ of size κ such that $M \cap L_\kappa$ is transitive and $M \cap \kappa = \alpha$ for some $\alpha \in S$. Let $f(\alpha)$ be the least ordinal such that α is singular in $L_{f(\alpha)}$ and let $r = (r_i)_{i < \omega}$ be some cofinal sequence in α in $L_{f(\alpha)}$. Now player **II** can play against τ in $L_{f(\alpha)}$ towards α using r . The replies of **I** will be in fact in M and the eventual sequences $(p_n^k)_{n < \omega}$, $k \in \{0, 1\}$, constructed by **I** will be in $L_{f(\alpha)}$ and so by definition $\bigcup_{i < \omega} p_n^k$ will be in T and so player **II** wins this game. \square

12.2 Claim. *If $S' \setminus S$ is ω -stationary, then F_S is not Borel-reducible to $F_{S'}$.*

Proof. The argument is as in [Kul13]. Suppose f is a Borel function from $[T]_\kappa$ to $[T]_\kappa$ which reduces F_S to $F_{S'}$ for some stationary S and S' such that $S' \setminus S$ is stationary. We will derive a contradiction. By Lemma 11 there exists a sequence $(D_i)_{i < \kappa}$ of dense open sets such that f is continuous on the co-meager set $D = \bigcap_{i < \kappa} D_i$. We will now define a strategy of player **I** in $G(T, S' \setminus S)$ such that if it is not a winning strategy, then the contradiction is achieved, so we are done by the claim above.

The strategy is as follows. At the first move, player **I** picks a function $\eta \in 2^\kappa$ with the property that both η and $1 - \eta$ are branches of T and in D . Since η and $1 - \eta$ are non-equivalent in F_S , $f(\eta)$ and $f(1 - \eta)$ are non-equivalent in $F_{S'}$. So there is a point α such that $f(\eta)(\alpha) \neq f(1 - \eta)(\alpha)$. Player **I** then finds α_0 such that $f[D \cap N_{\eta \upharpoonright \alpha_0}] \subset N_{f(\eta)(\alpha+1)}$ and $f[D \cap N_{(1-\eta) \upharpoonright \alpha_0}] \subset N_{f(1-\eta)(\alpha+1)}$. The first move is the pair (p_0^0, p_0^1) where $p_0^0 = \eta \upharpoonright \alpha_0$ and $p_0^1 = (1 - \eta) \upharpoonright \alpha_0$. Additionally player **I** keeps in mind the elements $q_0^0 = f(\eta) \upharpoonright (\alpha + 1)$ and $q_0^1 = f(1 - \eta) \upharpoonright (\alpha + 1)$. Suppose the players have played n moves and $(\beta_0, \dots, \beta_n)$ are the ordinals picked by player **II** and $((p_0^0, p_0^1), \dots, (p_n^0, p_n^1))$ the pairs picked by player **I**. Player **I** has also constructed a sequence $(q_i^0, q_i^1)_{i \leq n}$. If n is even, then

player **I** extends p_n^0 and p_n^1 into branches η and ξ of T such that $\eta(\alpha) = \xi(\alpha)$ implies $\alpha < \text{dom } p_n^0 = \text{dom } p_n^1 = \alpha_n$ and such that η and ξ are both in D . By the induction hypothesis $f(\eta)$ extends q_n^0 and $f(\xi)$ extends q_n^1 , so we can find $\beta'_n > \beta_n$ such that the continuations $q_{n+1}^0 = f(\eta) \upharpoonright \beta'_n$ and $q_{n+1}^1 = f(\xi) \upharpoonright \beta'_n$ are of equal length and for some $\beta \in \text{dom } q_{n+1}^0 \setminus \beta_n$ with $q_{n+1}^0(\beta) \neq q_{n+1}^1(\beta)$ (if such β'_n does not exist, then it implies that q_n^0 and q_n^1 cannot be extended to $F_{S'}$ -equivalent branches whereas p_n^0 and p_n^1 can be extended to F_S -equivalent branches, which would be a contradiction). Then player **I** finds an $\alpha_{n+1} > \beta'_n$ such that, denoting $p_{n+1}^0 = \eta \upharpoonright \alpha_{n+1}$ and $p_{n+1}^1 = \xi \upharpoonright \alpha_{n+1}$, we have

$$f[D \cap N_{p_{n+1}^0}] \subset N_{q_{n+1}^0}$$

and

$$f[D \cap N_{p_{n+1}^1}] \subset N_{q_{n+1}^1}.$$

The pair (p_{n+1}^0, p_{n+1}^1) is the next move. If n is odd, then player **I** proceeds in the same way, but with the only differences that now he picks η and ξ such that $\eta(\alpha) = \xi(\alpha)$ for all $\alpha > \text{dom } \alpha_n$ and finds q_{n+1}^0 and q_{n+1}^1 such that $q_{n+1}^0(\beta) = q_{n+1}^1(\beta)$ for some $\beta \in \text{dom } q_{n+1}^0 \setminus \beta_n$. This describes the strategy.

If player **II** beats this strategy in $G(T, S' \setminus S)$, it means that the limit of her moves, which is the same as the limit of the sequence $(\text{dom } p_n^i)_{n < \omega}$, $i \in \{0, 1\}$, is in S' and not in S . So by looking at the things that player **I** has constructed, we note that $p_\omega^0 = \bigcup_{n < \omega} p_n^0$ and $p_\omega^1 = \bigcup_{n < \omega} p_n^1$ can be extended to equivalent branches on the side of F_S , but $q_\omega^0 = \bigcup_{n < \omega} q_n^0$ and $q_\omega^1 = \bigcup_{n < \omega} q_n^1$ cannot be extended (in D) to equivalent branches on the range side $F_{S'}$ which is a contradiction, because $f[D \cap N_{p_\omega^i}] \subset N_{q_\omega^i}$, $i \in \{0, 1\}$. \square

To prove the Theorem, let $(S_i)_{i < \kappa}$ be a partition of S_ω^κ into disjoint stationary pieces. Then let \mathcal{A} be a maximal antichain in $\mathcal{P}(\kappa)$ (a set of size 2^κ of subsets of κ incomparable under inclusion) and define $G: \mathcal{A} \rightarrow \mathcal{E}$ by $G(A) = F_{\bigcup_{i \notin A} S_i}$. Then $G[\mathcal{A}]$ is an antichain by the claims above. Every element of this antichain is incomparable with identity: identity is not reducible to any of them, because of the small levels guaranteed by the weak Kurepa tree T . On the other hand any of the relations is not reducible to id because of the E_0 -component: the equivalence classes are dense in T which violates the continuity of any reduction even on an (arbitrary) co-meager set. \square

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